

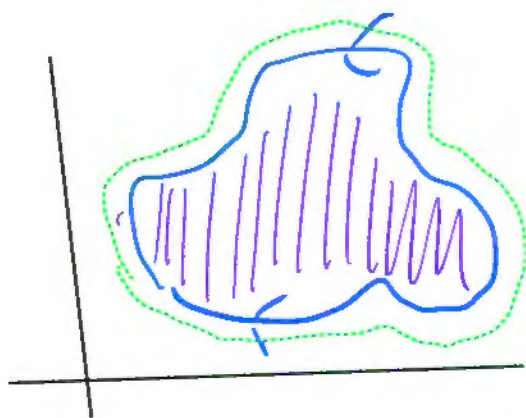
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Last Time: Green's Theorem.

Prop (Green's Theorem): Suppose

D is a region in the plane with its boundary a piecewise smooth simple-closed curve. If $P(x,y)$ and $Q(x,y)$ have cts partial derivatives on some open region, " R ", containing D , then

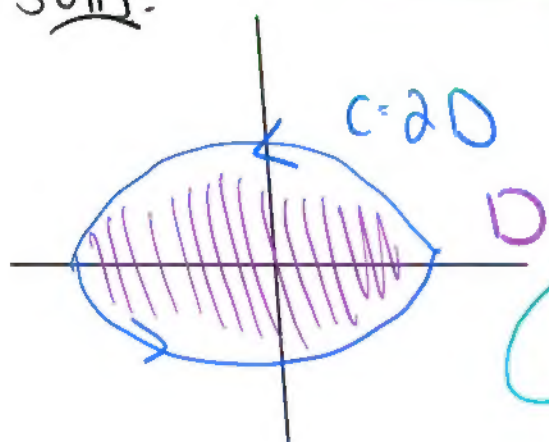
$$\left[\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \right]$$



Ex: Compute $\int_C y^4 dx + 2xy^2 dy$ for

C the positively oriented ellipse
 $x^2 + 2y^2 = 2$.

Soln:



Notice:

like a circle. So
make a substitution
to make it circle like
then use polar coords.

(Transformation)

$$\begin{cases} x = \sqrt{2} r \cos \theta \\ y = r \sin \theta \end{cases}$$

By Green's Theorem: $* 0 \leq r \leq 1 *$
 $0 \leq \theta \leq 2\pi$

$$\int_C y^4 dx + 2xy^2 dy$$

$$= \iint_D \left(\frac{\partial}{\partial x} [2xy^2] - \frac{\partial}{\partial y} [y^4] \right) dA$$

$$= \iint_D (2y^2 - 4y^3) dA$$

Notice cont.



unit disk

$$x^2 + 2y^2 = 2$$

$$\text{iff } (\sqrt{2} r \cos(\theta))^2 + 2(r \sin(\theta))^2 = 2$$

$$\text{iff } 2r^2 \cos^2(\theta) + 2r^2 \sin^2(\theta) = 2$$

$$\text{iff } 2r^2 (\cos^2 \theta + \sin^2 \theta) = 2$$

$$\text{iff } 2r^2 (\cos^2(\theta) + \sin^2(\theta)) = 2$$

$$\text{iff } r^2 = 1 \quad \text{iff } \underline{\underline{r=1 \text{ and } r \geq 0}}$$

Now compute Jacobian:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \det \begin{bmatrix} \sqrt{2} \cos \theta & -\sqrt{2} r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

$$= \sqrt{2} r \underbrace{\cos^2 \theta + \sin^2 \theta}_1$$

$$= \sqrt{2} r$$

$$\therefore \int_C y^4 dx + 2xy^2 dy = \iint_D (2y^2 - 4y^3) dA$$

$$= \int_{r=0}^1 \int_{\theta=0}^{2\pi} (2(r \sin \theta)^2 - 4(r \sin \theta)^3) \sqrt{2} r \, d\theta \, dr$$

$$= \int_{r=0}^1 \int_{\theta=0}^{2\pi} 2\sqrt{2} r^3 (\sin^2 \theta - 2r \sin^3 \theta) \, d\theta \, dr$$

$$= \int_{r=0}^1 2\sqrt{2} r^3 \int_{\theta=0}^{2\pi} \sin^2 \theta (1 - 2r \sin \theta) \, d\theta \, dr$$

$$= \int_{r=0}^1 2\sqrt{2} r^3 \int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) (1 - 2r \sin \theta) d\theta dr$$

eval. inner integral:

$$\int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) (1 - 2r \sin \theta) d\theta$$

$$= \int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) d\theta - 2r \int_{\theta=0}^{2\pi} (1 - \cos^2 \theta) \sin \theta d\theta$$

$u = \cos \theta$
 $du = -\sin \theta d\theta$

$$= \int_{\theta=0}^{2\pi} (1 - \frac{1}{2}(1 + \cos 2\theta)) d\theta - 2r \int_{\theta=0}^{2\pi} -(1 - u^2) du$$

$$= \int_{\theta=0}^{2\pi} (\frac{1}{2} - \frac{1}{2} \cos 2\theta) d\theta + 2r \left[u - \frac{1}{3} u^3 \right]_{\theta=0}^{2\pi}$$

$$= \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_{\theta=0}^{2\pi} + 2r \left[\cos \theta - \frac{1}{3} (\cos^3 \theta) \right]_{\theta=0}^{2\pi}$$

$$= \frac{1}{2} (2\pi - 0) - \frac{1}{4} (\sin(4\pi) - \sin(0)) + 2r ((\cos(2\pi) - \cos(0)) - \frac{1}{3} (\cos^3(2\pi) - \cos^3(0)))$$

$$= \pi - \frac{1}{4} \cdot 0 + 2r (0 - \frac{1}{3} \cdot 0) = \pi$$

eval outer integral:

$$\int_{r=0}^1 2\sqrt{2} r^3 \pi dr = \frac{2\sqrt{2}\pi}{4} [r^4]_{r=0}^1$$

$$= \frac{\pi}{\sqrt{2}} (1^4 - 0^4) = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_C y^4 dx + 2xy^2 dy = \boxed{\frac{\pi}{\sqrt{2}}}$$

NB: So far, all examples so far have turned line integrals into double integrals via Green's Theorem. BUT we can go in other way as well.

NB: If P & Q satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

then via Green's Theorem

$$\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D 1 dA = \text{Area}(D)$$

Ex: Compute the area of the general ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

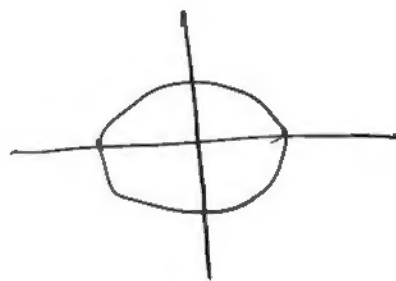
Soln: $\text{Area}(D) = \int_{\partial D} P dx + Q dy$

if we choose

$$P, Q \text{ w/ } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

e.g. $Q=0, P=-y$: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}[0] - \frac{\partial}{\partial y}[-y] = 1$

$$\therefore \text{area}(D) = \int_{\partial D} -y dx + 0 dy = - \int_{\partial D} y dx$$



find 2D:

the ellipse 2D is parameterized by

$$\vec{r}(t) = \langle a \cos(t), b \sin(t) \rangle \text{ on}$$

$$0 \leq t \leq 2\pi \quad \therefore dx = x'(t) dt$$

$$= - \int_{t=0}^{2\pi} b \sin(t) \cdot -a \sin(t) dt$$

$$= ab \int_{t=0}^{2\pi} \sin^2(t) dt$$

$$= ab \int_{t=0}^{2\pi} \frac{1}{2} (1 - \cos(2t)) dt$$

$$= \frac{1}{2} ab \left[t - \frac{1}{2} \sin(2t) \right]_{t=0}^{2\pi}$$

$$= \frac{1}{2} ab \left((2\pi - 0) - \frac{1}{2} (0 - 0) \right) = \boxed{ab\pi}$$

§16.5: Curl and Divergence

Goal: Define and study two new operations on vector fields.

Curl: The curl of vector field \vec{v} on \mathbb{R}^3 is
 $\vec{v} = \langle P, Q, R \rangle$

$$\text{Curl}(\vec{v}) = " \nabla \times \vec{v} " = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Ex: Compute $\text{curl}(\vec{v})$ for $\vec{v} = \langle xy, xyz, -y^2 \rangle$

Soln:

$$\text{curl}(\vec{v}) = \vec{v} \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz & -y^2 \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[-y^2] - \frac{\partial}{\partial z}[xyz], -\left[\frac{\partial}{\partial x}[-y^2] - \frac{\partial}{\partial z}[xy]\right], -\frac{\partial}{\partial y}[xy] \right\rangle$$

$$= \langle -2y - xy, -(0 - 0), yz - x \rangle$$

$$= \langle -xy - 2y, 0, yz - x \rangle$$

Observation: Suppose $\vec{v} = \nabla f$ is conservative.

i.e. $\vec{v} = \langle f_x, f_y, f_z \rangle$

$$\text{So, } \text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{bmatrix}$$

$$= \left\langle \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, -\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right), \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right\rangle$$

$$\begin{aligned}
 &= \langle \overbrace{f_{zy} - f_{yz}}^{\quad}, -(\overbrace{f_{zx} - f_{xz}}^{\quad}), \overbrace{f_{yx} - f_{xy}}^{\quad} \rangle \\
 &= \langle 0, 0, 0 \rangle = \vec{0} \text{ by Clairaut's theorem.}
 \end{aligned}$$

Point: $\text{Curl}(\nabla f) = \vec{0}$, i.e. Curl of a conservative ∇f is $\vec{0}$.

~~Prop~~: A vector field w/ components having cts. partial derivatives is conservative if and only if $\text{Curl}(\vec{v}) = \vec{0}$.

Divergence: The divergence of vector field \vec{v} , $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\begin{aligned}
 \text{div}(\vec{v}) &= " \nabla \cdot \vec{v} " = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right\rangle \cdot \langle v_1, v_2, \dots, v_n \rangle \\
 &= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \dots + \frac{\partial v_n}{\partial x_n}
 \end{aligned}$$

Ex: For $\vec{v} = \langle xy, xy^2, -y^2 \rangle$ compute $\text{div}(\vec{v})$.

$$\begin{aligned}\text{Soln. } \text{div}(\vec{v}) &= \frac{\partial}{\partial x} [xy] + \frac{\partial}{\partial y} [xy^2] + \frac{\partial}{\partial z} [-y^2] \\ &= y + x^2 + 0 = y + x^2\end{aligned}$$

Suppose: $\vec{v} = \text{curl}(\vec{w})$ for $w = \langle P, Q, R \rangle$

$$\vec{v} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$$

$$\text{now, } \text{div}(\vec{v}) = \frac{\partial}{\partial x} [R_y - Q_z] + \frac{\partial}{\partial y} [-(R_x - P_z)] + \frac{\partial}{\partial z} [Q_x - P_y]$$

$$= (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) + (R_{yx} - R_{xy})$$

$$= 0 + 0 + 0 = 0 \quad \text{by Clairaut's theorem.}$$

Prop: A vector field is the curl of another vector field if and only if its divergence is zero.

NB: above we should $\text{div}(\text{curl}(\vec{v})) = 0$.